

# Discrete Symmetric Planar Dynamics

B. Alarcón, S. B. S. D. Castro and I. S. Labouriau

**Abstract** We review previous results providing sufficient conditions to determine the global dynamics for equivariant maps of the plane with a unique fixed point which is also hyperbolic.

## 1 Introduction

The Discrete Markus-Yamabe Question is a problem concerning discrete dynamics, formulated in dimension  $n$  by Cima *et al.* [8] as follows:

**[DMYQ( $n$ )]** *Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a  $C^1$  map such that  $f(0) = 0$  and for any  $x \in \mathbf{R}^n$ ,  $Jf(x)$  has all its eigenvalues with modulus less than one. Is it true that 0 is a global attractor for the discrete dynamical system generated by  $f$ ?*

It is known that the answer is affirmative in dimension 1 and there are counter-examples for dimensions higher than 2, see Cima *et al.* [9] and van den Essen and Hubbers [10].

In dimension 2, Cima *et al.* [8] prove that an affirmative answer is obtained when  $f$  is a polynomial map, and provide a counter example which is a rational map. After this, research on planar maps focused on the quest for minimal sufficient conditions under which the DMYQ has an affirmative answer. Alarcón *et al.* [6] use the

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existence of an invariant embedded curve joining the origin to infinity to show the global stability of the origin. Symmetry is a natural context for the existence of such a curve, and this led us to a symmetric approach to this problem and to the results in [1, 2, 3, 4, 5] that we review in this article.

The present article studies maps  $f$  of the plane which preserve symmetries described by the action of a compact Lie group. In this setting we characterise the possible local dynamics near the unique fixed point of  $f$ , that we assume hyperbolic. We establish for which symmetry groups local dynamics extends globally. For the remaining groups we present illustrative examples.

## 2 Preliminaries

This section consists of definitions and known results about topological dynamics and equivariant theory. These are grouped in two separate subsections, which are elementary for readers in each field, containing material from the corresponding sections of [1, 2, 3, 4, 5] and is included here for ease of reference.

### 2.1 Topological Dynamics

We consider planar topological embeddings, that is, continuous and injective maps defined in  $\mathbf{R}^2$ . The set of topological embeddings of the plane is denoted by  $\text{Emb}(\mathbf{R}^2)$ .

Recall that for  $f \in \text{Emb}(\mathbf{R}^2)$  the equality  $f(\mathbf{R}^2) = \mathbf{R}^2$  may not hold. Since every map  $f \in \text{Emb}(\mathbf{R}^2)$  is open (see [13]), we will say that  $f$  is a homeomorphism if  $f$  is a topological embedding defined onto  $\mathbf{R}^2$ . The set of homeomorphisms of the plane will be denoted by  $\text{Hom}(\mathbf{R}^2)$ . When  $\mathcal{H}$  is one of these sets we denote by  $\mathcal{H}^+$  (and  $\mathcal{H}^-$ ) the subset of orientation preserving (reversing) elements of  $\mathcal{H}$ .

We denote by  $\text{Fix}(f)$  the set of fixed points of a continuous map  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ .

Let  $\omega(p)$  be the set of points  $q$  for which there is a sequence  $n_j \rightarrow +\infty$  such that  $f^{n_j}(q) \rightarrow p$ . If  $f \in \text{Hom}(\mathbf{R}^2)$  then  $\alpha(p)$  denotes the set  $\omega(p)$  under  $f^{-1}$ .

Let  $f \in \text{Emb}(\mathbf{R}^2)$  and  $p \in \mathbf{R}^2$ . We say that  $\omega(p) = \infty$  if  $\|f^n(p)\| \rightarrow \infty$  as  $n$  goes to  $\infty$ . Analogously, if  $f \in \text{Hom}(\mathbf{R}^2)$ , we say that  $\alpha(p) = \infty$  if  $\|f^{-n}(p)\| \rightarrow \infty$  as  $n$  goes to  $\infty$ .

We say that a map  $f \in \text{Emb}(\mathbf{R}^2)$  is *dissipative* if there exists a compact set  $W \subset \mathbf{R}^2$  that is positively invariant and attracts uniformly all compact sets. This means that  $f(W) \subset W$  and for each  $x \in \mathbf{R}^2$ ,

$$\text{dist}(f^n(x), W) \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

uniformly on balls  $|x| \leq r$ ,  $r > 0$ . Observe that in the case  $f \in \text{Hom}(\mathbf{R}^2)$  the dissipativity of  $h$  means that  $\infty$  is asymptotically stable for  $f^{-1}$ .

We say that  $0 \in \text{Fix}(f)$  is a *local attractor* if its basin of attraction  $\mathcal{U} = \{p \in \mathbf{R}^2 : \omega(p) = \{0\}\}$  contains an open neighbourhood of 0 in  $\mathbf{R}^2$  and that 0 is a *global attractor* if  $\mathcal{U} = \mathbf{R}^2$ . The origin is a *stable fixed point* if for every neighborhood  $U$  of 0 there exists another neighborhood  $V$  of 0 such that  $f(V) \subset V$  and  $f(V) \subset U$ . Therefore, the origin is an *asymptotically local (global) attractor* or a *(globally) asymptotically stable fixed point* if it is a stable local (global) attractor. See [7] for examples.

We say that  $0 \in \text{Fix}(f)$  is a *local repeller* if there exists a neighbourhood  $V$  of 0 such that  $\omega(p) \notin V$  for all  $0 \neq p \in \mathbf{R}^2$  and a *global repeller* if this holds for  $V = \mathbf{R}^2$ .

We say that the origin is an *asymptotically global repeller* if it is a global repeller and, moreover, if for any neighbourhood  $U$  of 0 there exists another neighbourhood  $V$  of 0, such that,  $V \subset f(V)$  and  $V \subset f(U)$ .

When the origin is a fixed point of a  $C^1$ -map of the plane we say the origin is a *local saddle* if the two eigenvalues of  $Df_0$ ,  $\alpha, \beta$ , are both real and verify  $0 < |\alpha| < 1 < |\beta|$ . In case the two eigenvalues are strictly positive we say the origin is a *direct saddle*. We say that the origin is a *global (topological) saddle* for a  $C^1$ -homeomorphism if additionally its stable and unstable manifolds  $W^s(0, f)$ ,  $W^u(0, f)$  are unbounded sets that do not accumulate on each other, except at 0 and  $\infty$ , and such that

$$\mathbf{R}^2 \setminus (W^s \cup W^u \cup \{\infty\}) = U_1 \cup U_2 \cup U_3 \cup U_4,$$

where for all  $i = 1, \dots, 4$   $U_i \subset \mathbf{R}^2$  is open connected and homeomorphic to  $\mathbf{R}^2$  verifying:

- i) either  $f(U_i) = U_i$  or there exists an involution  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  such that  $(f \circ \varphi)(U_i) = U_i$
- ii) for all  $p \in U_i$  both  $\|f^n(p)\| \rightarrow \infty$  and  $\|f^{-n}(p)\| \rightarrow \infty$  as  $n$  goes to  $\infty$ .

We say that  $f \in \text{Emb}(\mathbf{R}^2)$  has *trivial dynamics* if  $\omega(p) \subset \text{Fix}(f)$ , for all  $p \in \mathbf{R}^2$ . Moreover, we say that a planar homeomorphism has trivial dynamics if both  $\omega(p), \alpha(p) \subset \text{Fix}(f)$ , for all  $p \in \mathbf{R}^2$ .

Let  $f : \mathbf{R}^N \rightarrow \mathbf{R}^N$  be a continuous map. Let  $\gamma : [0, \infty) \rightarrow \mathbf{R}^2$  be a topological embedding of  $[0, \infty)$ . As usual, we identify  $\gamma$  with  $\gamma([0, \infty))$ . We will say that  $\gamma$  is an *f-invariant ray* if  $\gamma(0) = (0, 0)$ ,  $f(\gamma) \subset \gamma$ , and  $\lim_{t \rightarrow \infty} |\gamma(t)| = \infty$ , where  $|\cdot|$  denotes the usual Euclidean norm.

**Proposition 1 (Alarcón et al. [6]).** *Let  $f \in \text{Emb}^+(\mathbf{R}^2)$  be such that  $\text{Fix}(f) = \{0\}$ . If there exists an f-invariant ray  $\gamma$ , then  $f$  has trivial dynamics.*

**Corollary 1.** *Let  $f \in \text{Hom}^+(\mathbf{R}^2)$  be such that  $\text{Fix}(f) = \{0\}$ . If there exists an f-invariant ray  $\gamma$ , then for each  $p \in \mathbf{R}^2$ , as  $n$  goes to  $\pm\infty$ , either  $f^n(p)$  goes to 0 or  $\|f^n(p)\| \rightarrow \infty$ .*

In order to explain the construction of examples in Section 5 we need to introduce the concept of prime end.

We say that  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is an *admissible homeomorphism* if  $f$  is orientation preserving, dissipative and has an asymptotically stable fixed point with proper and

unbounded basin of attraction  $U \subset \mathbf{R}^2$ . Note that  $U$  is non empty, so the proper condition follows when the fixed point is not a global attractor. Since  $f(U) = U$ , we can obtain automatically the unboundedness condition if we suppose that  $f$  is area contracting.

Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be an admissible homeomorphism and consider the compactification of  $f$  to the Riemann sphere  $f : \mathbf{S}^2 \rightarrow \mathbf{S}^2$ . Hence  $U \subset \mathbf{S}^2 = \mathbf{R}^2 \cup \{\infty\}$ . A *crosscut*  $C$  of  $U$  is an arc homeomorphic to the segment  $[0, 1]$  such that  $a, b \notin U$  and  $\dot{C} = C \setminus \{a, b\} \subset U$ , where  $a$  and  $b$  are the extremes of  $C$ . Every crosscut divides  $U$  into two connected components homeomorphic to the open disk  $\mathbf{d} = \{z \in \mathbf{C} : |z| < 1\}$ .

Let  $x^*$  be a point in  $U$ . For convenience we will consider only the crosscut such that  $x^* \notin C$ . We denote by  $D(C)$  the component of  $U \setminus C$  that does not contain  $x^*$ . A *null-chain* is a sequence of pairwise disjoint crosscuts  $\{C_n\}_{n \in \mathbf{N}}$  such that

$$\lim_{n \rightarrow \infty} \text{diam}(C_n) = 0 \text{ and } D(C_{n+1}) \subset D(C_n),$$

where  $\text{diam}(C_n)$  is the diameter of  $C_n$  on the Riemann sphere.

Two *null-chains* are *equivalent*  $\{C_n\}_{n \in \mathbf{N}} \sim \{C_n^*\}_{n \in \mathbf{N}}$  if given  $m \in \mathbf{N}$

$$D(C_n) \subset D(C_m^*) \text{ and } D(C_n^*) \subset D(C_m),$$

for  $n$  large enough. A *prime end* is defined as a class of equivalence of a null-chain and the space of prime ends is

$$\mathbf{P} = \mathbf{P}(U) = \mathcal{C} / \sim,$$

where  $\mathcal{C}$  is the set of all null-chains of  $U$ .

The disjoint union  $U^* = U \cup \mathbf{P}$  is a topological space homeomorphic to the closed disk  $\bar{\mathbf{d}} = \{z \in \mathbf{C} : |z| \leq 1\}$  such that its boundary is precisely  $\mathbf{P}$ .

It is well studied in [14] that the Theory of Prime Ends implies that an admissible homeomorphism  $f$  induces an orientation preserving homeomorphism  $f^* : \mathbf{P} \rightarrow \mathbf{P}$  in the space of prime ends. This topological space is homeomorphic to the circle, that is  $\mathbf{P} \simeq \mathbf{T}$ , and hence the rotation number of  $f^*$  is well defined, say  $\bar{\rho} \in \mathbf{T}$ . The *rotation number* for an admissible homeomorphism is defined by  $\rho(f) = \bar{\rho}$ .

## 2.2 Equivariant Theory

Let  $\Gamma$  be a compact Lie group acting on  $\mathbf{R}^2$ , that is, a group which has the structure of a compact  $C^\infty$ -differentiable manifold such that the map  $\Gamma \times \Gamma \rightarrow \Gamma$ ,  $(x, y) \mapsto xy^{-1}$  is of class  $C^\infty$ . The following definitions and results are taken from Golubitsky *et al.* [11], especially Chapter XII, to which we refer the reader interested in further detail.

We think of a group mostly through its action or representation on  $\mathbf{R}^2$ . A *linear action* of  $\Gamma$  on  $\mathbf{R}^2$  is a continuous mapping

$$\begin{aligned}\Gamma \times \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ (\gamma, p) &\mapsto \gamma p\end{aligned}$$

such that, for each  $\gamma \in \Gamma$  the mapping  $p_\gamma$  that takes  $p$  to  $\gamma p$  is linear and, given  $\gamma_1, \gamma_2 \in \Gamma$ , we have  $\gamma_1(\gamma_2 p) = (\gamma_1 \gamma_2)p$ . Furthermore the identity in  $\Gamma$  fixes every point. The mapping  $\gamma \mapsto p_\gamma$  is called the *representation* of  $\Gamma$  and describes how each element of  $\Gamma$  transforms the plane.

We consider only standard group actions and representations. A representation of a group  $\Gamma$  on a vector space  $V$  is *absolutely irreducible* if the only linear mappings on  $V$  that commute with  $\Gamma$  are scalar multiples of the identity.

Given a map  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ , we say that  $\gamma \in \Gamma$  is a *symmetry* of  $f$  if  $f(\gamma x) = \gamma f(x)$ . We define the *symmetry group* of  $f$  as the biggest closed subset of  $GL(2)$  containing all the symmetries of  $f$ . It will be denoted by  $\Gamma_f$ .

We say that  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is  $\Gamma$ -*equivariant* or that  $f$  *commutes* with  $\Gamma$  if

$$f(\gamma x) = \gamma f(x) \quad \text{for all } \gamma \in \Gamma.$$

It follows that every map  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is equivariant under the action of its symmetry group, that is,  $f$  is  $\Gamma_f$ -equivariant.

Let  $\Sigma$  be a subgroup of  $\Gamma$ . The *fixed-point subspace* of  $\Sigma$  is

$$\text{Fix}(\Sigma) = \{p \in \mathbf{R}^2 : \sigma p = p \text{ for all } \sigma \in \Sigma\}.$$

If  $\Sigma$  is generated by a single element  $\sigma \in \Gamma$ , we write  $\text{Fix}(\sigma)$ .

We note that, for each subgroup  $\Sigma$  of  $\Gamma$ ,  $\text{Fix}(\Sigma)$  is invariant by the dynamics of a  $\Gamma$ -equivariant map ([11], XIII, Lemma 2.1).

When  $f$  is  $\Gamma$ -equivariant, we can use the symmetry to generalize information obtained for a particular point. This is achieved through the *group orbit*  $\Gamma x$  of a point  $x$ , which is defined to be

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}.$$

**Lemma 1.** *Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be  $\Gamma$ -equivariant and let  $p$  be a fixed point of  $f$ . Then all points in the group orbit of  $p$  are fixed points of  $f$ .*

*Proof.* If  $f(p) = p$  it follows that  $f(\gamma p) = \gamma f(p) = \gamma p$ , showing that  $\gamma p$  is a fixed point of  $f$  for all  $\gamma \in \Gamma$ .

The relation between the group action and the Jacobian matrix of an equivariant map  $f$  is obtained through the following

**Lemma 2.** *Let  $f : V \rightarrow V$  be a  $\Gamma$ -equivariant map differentiable at the origin. Then  $Df(0)$ , the Jacobian of  $f$  at the origin, commutes with  $\Gamma$ .*

*Proof.* Since  $f$  is  $\Gamma$ -equivariant we have  $f(\gamma.v) = \gamma f(v)$  for all  $\gamma \in \Gamma$  and  $v \in V$ . Differentiating both sides of the equality with respect to  $v$ , we obtain  $Df(\gamma.v)\gamma = \gamma Df(v)$  and, evaluating at the origin gives  $Df(0)\gamma = \gamma Df(0)$ .

### 3 Symmetries in the plane

In this section, we describe the consequences for the local dynamics arising from the fact that a map is equivariant under the action of a compact Lie group  $\Gamma$ . These are patent in the structure of the Jacobian matrix at the origin, obtained using Lemma 2.

Since every compact Lie group in  $GL(2)$  can be identified with a subgroup of the orthogonal group  $O(2)$ , we need only be concerned with the groups we list below.

Compact subgroups of  $O(2)$

- $O(2)$ , acting on  $\mathbf{R}^2 \simeq \mathbf{C}$  as the group generated by  $\theta$  and  $\kappa$  given by

$$\theta.z = e^{i\theta}z, \quad \theta \in S^1 \quad \text{and} \quad \kappa.z = \bar{z}.$$

- $SO(2)$ , acting on  $\mathbf{R}^2 \simeq \mathbf{C}$  as the group generated by  $\theta$  given by

$$\theta.z = e^{i\theta}z, \quad \theta \in S^1.$$

- $D_n, n \geq 2$ , acting on  $\mathbf{R}^2 \simeq \mathbf{C}$  as the finite group generated by  $\zeta$  and  $\kappa$  given by

$$\zeta.z = e^{\frac{2\pi i}{n}}z \quad \text{and} \quad \kappa.z = \bar{z}.$$

- $Z_n, n \geq 2$ , acting on  $\mathbf{R}^2 \simeq \mathbf{C}$  as the finite group generated by  $\zeta$  given by

$$\zeta.z = e^{\frac{2\pi i}{n}}z.$$

- $Z_2(\langle \kappa \rangle)$ , acting on  $\mathbf{R}^2$  as

$$\kappa.(x, y) = (x, -y).$$

Since most of our results depend on the existence of a unique fixed point for  $f$ , it is worthwhile noting that the group actions we are concerned with are such that  $\text{Fix}(\Gamma) = \{0\}$ . Therefore, if  $f$  is  $\Gamma$ -equivariant then  $f(0) = 0$ .

If the representation is absolutely irreducible, we know that  $Df(0)$  is a multiple of the identity and thus it has one real eigenvalue of geometric multiplicity two. Therefore, the origin is locally either an attractor or a repeller. We have the following

**Lemma 3.** *The standard representation on  $\mathbf{R}^2$  is absolutely irreducible for  $O(2)$  and  $D_n$  with  $n \geq 3$  and for no other subgroup of  $O(2)$ .*

*Proof.* The proof follows by direct computation.

- $O(2)$ : the generators of this group are  $\theta$  and  $\kappa$  and it suffices to find the linear matrices that commute with both. A real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

commutes with  $\kappa$  if and only if  $b = c = 0$ . In order for such a matrix to commute with any rotation it must be

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

which holds when  $a = d$  or  $\sin \theta = 0$  for all  $\theta \in S^1$ . Therefore, the action of  $O(2)$  is absolutely irreducible.

- $SO(2)$ : the elements of  $SO(2)$  are rotation matrices which commute with any other rotation matrix, also non-diagonal ones.
- $D_n$ ,  $n \geq 3$ : see the proof for  $O(2)$ . In the last step, we must have  $a = d$  or  $\sin 2\pi i/n = 0$  which is never satisfied for  $n \geq 3$ . Hence, the action is absolutely irreducible.
- $\mathbf{Z}_n$ ,  $n \geq 3$ : as for  $SO(2)$ , any rotation matrix commutes with the rotation of  $2\pi/n$ , including non-diagonal ones.
- $\mathbf{Z}_2(\langle \kappa \rangle)$ : see the proof for  $\kappa \in O(2)$  to conclude that linear commuting matrices are diagonal but not necessarily linear multiples of the identity.
- $\mathbf{Z}_2$ : all linear maps commute with  $-Id$ .
- $D_2 = \mathbf{Z}_2 \oplus \mathbf{Z}_2(\langle \kappa \rangle)$ : as above,  $\mathbf{Z}_2$  introduces no restrictions and for commuting with  $\kappa$  it suffices that the map is diagonal.

The following result is then a straightforward consequence of the previous proof.

**Lemma 4.** *The linear maps that commute with the standard representations of the subgroups of  $O(2)$  are rotations and homotheties (and their compositions) for  $SO(2)$  and  $\mathbf{Z}_n$ ,  $n \geq 3$ , linear multiples of the identity for  $O(2)$  and  $D_n$ ,  $n \geq 3$ , any linear map for  $\mathbf{Z}_2$  and maps represented by diagonal matrices for the remaining groups.*

*Proof.* The only linear maps that were not already explicitly calculated in the previous proof are those that commute with rotations. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if and only if either  $\sin \theta = 0$  for all  $\theta \in S^1$  or  $a = d$  and  $b = -c$ . Hence, the only maps commuting with either  $SO(2)$  or  $\mathbf{Z}_n$ ,  $n \geq 3$ , are rotations and homotheties and their compositions.

With the results obtained so far, we are able to describe the Jacobian matrix at the origin for maps equivariant under each of the groups above.

**Proposition 2 (Proposition 2.3 in [4]).** *Let  $f$  be a planar map differentiable at the origin. The admissible forms for the Jacobian matrix of  $f$  at the origin are those given in Table 1 depending on the symmetry group of  $f$ .*

Furthermore, the symmetry constrains the normal form as described in [3, Theorem 2.1] and in the next result, and its consequences for the linear part of  $f$  appear in Table 1.

Symmetry group	$Df(0)$	hyperbolic local dynamics
$O(2)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \alpha \in \mathbf{R}$	attractor / repellor
$SO(2)$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \alpha, \beta \in \mathbf{R}$	attractor / repellor
$D_n, n \geq 3$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \alpha \in \mathbf{R}$	attractor / repellor
$\mathbf{Z}_n, n \geq 3$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \alpha, \beta \in \mathbf{R}$	attractor / repellor
$\mathbf{Z}_2$	any matrix	saddle / attractor / repellor
$\mathbf{Z}_2(\langle \kappa \rangle)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \alpha, \beta \in \mathbf{R}$	saddle / attractor / repellor
$D_2 = \mathbf{Z}_2 \oplus \mathbf{Z}_2(\langle \kappa \rangle)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \alpha, \beta \in \mathbf{R}$	saddle / attractor / repellor

**Table 1** Compact subgroups of  $O(2)$  and the admissible forms of the Jacobian at the origin of maps equivariant under the standard action of each group. If in addition the Jacobian at the origin is hyperbolic, then this determines the local stability.

**Proposition 3 (Proposition 3.1 in [3]).** *Let  $\Gamma$  be a compact Lie group acting on  $\mathbf{R}^2$ . Assume  $\Gamma$  is the symmetry group of a polynomial map  $f$ .*

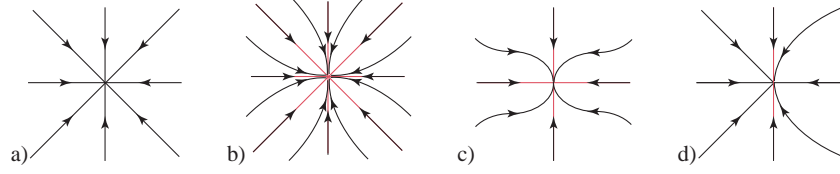
(i) *If  $\kappa \in \Gamma$ , then  $f$  does not answer the DMYQ(2) in the affirmative unless  $f$  is of the form:*

$$f(x, y) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + y^2 p(y^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) *If there is an element  $\zeta \in \Gamma$  of order  $n \geq 3$ , then  $f$  does not answer the DMYQ(2) in the affirmative unless  $f$  is linear. Moreover, the linear part of  $f$  is either a homothety or a rotation matrix.*



## 4 Dynamics — local to global



**Fig. 1** Local/global attractor with symmetry: a)  $O(2)$ ; b)  $D_4$  (without symmetries  $\mathbf{Z}_8$  or  $SO(2)$ ); c)  $D_2$  (without symmetry  $D_4$ ); d)  $\mathbf{Z}_2\langle\kappa\rangle$  (without symmetry  $D_4$ )

Figure 1 illustrates the dynamics near the origin of equivariant maps for several symmetry groups. A common feature of Figures 1 a)–d) is the existence of at least one symmetry axis. This axis is the subspace fixed by a reflection and hence it is invariant under the dynamics. Such a fixed-point subspace naturally contains an invariant ray (see [4, Lemma 3.3]). This allows us to use Proposition 1 to obtain the following results:

**Proposition 4 (Proposition 3.4 in [4]).** *Let  $f \in \text{Emb}(\mathbf{R}^2)$  have symmetry group  $\Gamma$  with  $\kappa \in \Gamma$ , such that  $\text{Fix}(f) = \{0\}$ . Suppose one of the following holds:*

- a)  $f \in \text{Emb}^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus \text{Fix}\langle\kappa\rangle$ .
- b)  $\text{Fix}(f^2) = \{0\}$ .

*Then for each  $p \in \mathbf{R}^2$  either  $\omega(p) = \{0\}$  or  $\omega(p) = \infty$ .*

The next example shows that assumption b) in Proposition 4 is necessary in the case where  $f$  interchanges connected components of  $\mathbf{R}^2 \setminus \text{Fix}\langle\kappa\rangle$ .

Example:

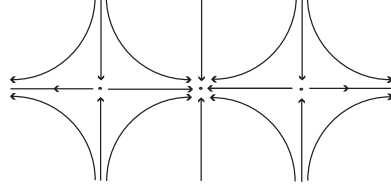
Consider the map  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  defined by

$$f(x, y) = \left( -ax^3 + (a-1)x, -\frac{y}{2} \right) \quad 0 < a < 1.$$

It is easily checked that  $f$  has symmetry group  $D_2$  and verifies (see Figure 2):

1.  $f \in \text{Emb}^+(\mathbf{R}^2)$  is an orientation-preserving diffeomorphism.
2.  $\text{Spec}(f) \cap [0, \infty) = \emptyset$ .
3.  $0$  is a local hyperbolic attractor.
4.  $\text{Fix}(f^2) \neq \{0\}$ .

**Fig. 2** A local attractor which is not a global attractor due to the existence of periodic orbits.



**Theorem 1 (Theorem 3.5 in [4]).** Let  $f \in \text{Emb}(\mathbf{R}^2)$  be dissipative with symmetry group  $\Gamma$  with  $\kappa \in \Gamma$  such that  $\text{Fix}(f) = \{0\}$ . Suppose in addition that one of the following holds:

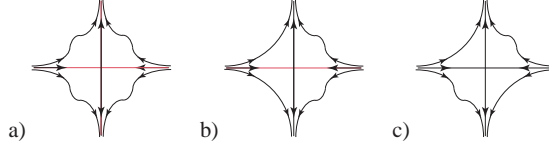
- a)  $f \in \text{Emb}^+(\mathbf{R}^2)$  and  $f$  does not interchange connected components of  $\mathbf{R}^2 \setminus \text{Fix}(\kappa)$ .
- b) There exist no 2-periodic orbits.

Then 0 is a global attractor.

**Corollary 2 (Corollary 3.6 in [4]).** Suppose the assumptions of Theorem 1 are verified and  $f$  is differentiable at 0. If every eigenvalue of  $Df(0)$  has norm strictly less than one, then 0 is a global asymptotic attractor.

For analogous results concerning a repellor see [4].

**Fig. 3** Local/global saddle with symmetry: a)  $D_2$ ; b)  $\mathbf{Z}_2(\kappa)$ ; c)  $\mathbf{Z}_2$ .



For the groups  $\mathbf{Z}_2$ ,  $\mathbf{Z}_2(\langle \kappa \rangle)$  and  $D_2 = \mathbf{Z}_2 \oplus \mathbf{Z}_2(\langle \kappa \rangle)$  the origin may also be a saddle as illustrated in Figure 3. For  $D_2$ , we have:

**Proposition 5 ([5]).** Let  $f \in \text{Hom}(\mathbf{R}^2)$  be a  $C^1$ -homeomorphism with symmetry group  $D_2$  such that  $\text{Fix}(f) = \{0\}$ . Suppose also that one of the following holds:

- a)  $f$  is orientation preserving and 0 is a direct saddle.
- b) There exist no 2-periodic orbits.

Then if 0 is a local saddle, then 0 is a global saddle.

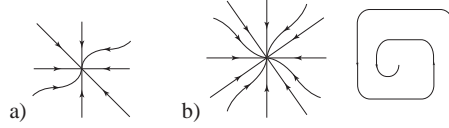
In order to obtain a global saddle for  $f$  with symmetry group either  $\mathbf{Z}_2$  or  $\mathbf{Z}_2(\langle \kappa \rangle)$ , we need the additional assumption that  $f$  is a diffeomorphism, see [5].

## 5 Strictly Local Dynamics

Figure 4 shows the local dynamics for maps equivariant under the action of groups that do not contain a reflection. These are  $SO(2)$  and  $\mathbf{Z}_n$ . For these groups, local dynamics of attractor/repellor type does not necessarily extend to global dynamics, as we proceed to indicate.

We use examples referring to a local attractor, examples with a local repellor may be obtained considering  $f^{-1}$ .

**Fig. 4** Local attractor with symmetry: a)  $\mathbf{Z}_2$ ; b)  $\mathbf{Z}_4$ .



The dynamics of an  $SO(2)$ -symmetric embedding is mostly determined by its radial component, as can be seen by writing  $f$  in polar coordinates as  $f(\rho, \theta) = (R(\rho, \theta), T(\rho, \theta))$ . It is easily shown that since  $f$  is  $SO(2)$ -equivariant, the radial component  $R(\rho, \theta)$  only depends on  $\rho$  and  $R \in Emb(\mathbf{R}^+)$ . The fixed points of the radial component are invariant circles for  $f$  hence knowledge about local dynamics does not contribute to the description of global dynamics unless  $\text{Fix}(R) = \{0\}$ .

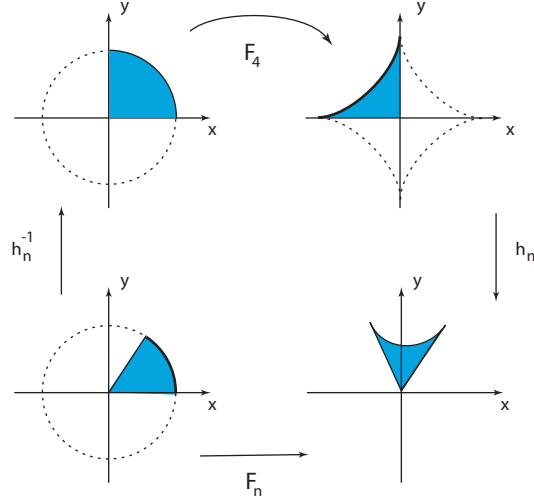
The next two theorems show how a local attractor may be prevented from being a global attractor in a  $\mathbf{Z}_n$ -equivariant problem. Thus the examples in Figures 4 a) and b) may or may not extend to the whole plane.

**Theorem 2 (Theorem 3.1 in [2]).** *For each  $n \geq 2$  there exists  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  such that:*

- a)  $f$  is a differentiable homeomorphism;
- b)  $f$  has symmetry group  $\mathbf{Z}_n$ ;
- c)  $\text{Fix}(f) = \{0\}$ ;
- d) The origin is a local attractor;
- e) There exists a periodic orbit of minimal period  $n$ .

The idea of the proof is to start with a  $\mathbf{Z}_4$ -equivariant example due to Szlenk (see [8]), sharing the same properties. Each quadrant of the plane is invariant under the map  $f_4$  of this example. We deform the first quadrant into a sector of the plane, of angle  $2\pi/n$  and then use the  $\mathbf{Z}_n$  symmetry to cover the rest of the plane, as illustrated in Figure 5. The main difficulty is to prove that the result is a differentiable homeomorphism.

The  $\mathbf{Z}_m$ -equivariant homeomorphisms constructed in Theorem 2 have rotation number  $1/m$ . So we might be led to think that the presence of the  $\mathbf{Z}_m$ -symmetry implies that the rotation number of the homeomorphism should be rational. One



**Fig. 5** Construction of the  $\mathbf{Z}_n$ -equivariant example  $F_n$  in a fundamental domain of the  $\mathbf{Z}_n$ -action, shown here for  $n = 6$ .

consequence would be that the asymptotically stable fixed point is a global attractor if and only if there are no periodic points different from the fixed point.

The next result shows that this is false. We prove in [1] the existence of  $\mathbf{Z}_m$ -equivariant and dissipative homeomorphisms in the plane with an asymptotically stable fixed point such that the induced map in the space of prime ends is conjugated to a Denjoy map, which is also  $\mathbf{Z}_m$ -equivariant. The idea is to construct  $\mathbf{Z}_m$ -equivariant Denjoy maps in the circle and then, in the context of symmetry, to reproduce the construction used to prove the following:

**Proposition 6 (Proposition 2 in [12]).** *Given  $w \in (0, 1) \setminus \mathbf{Q}$  and a Denjoy map  $\phi$ , there exists an admissible map  $f$  with rotation number  $\bar{w}$  and such that  $f^*$  is topologically conjugated to  $\phi$ .*

Observe that two admissible homeomorphisms  $f_1, f_2$  with the same basin of attraction  $U$  verify that

$$(f_1 \circ f_2)^* = f_1^* \circ f_2^*.$$

Let  $f$  be an admissible homeomorphisms with basin of attraction  $U$ . Suppose  $f$  is  $\mathbf{Z}_m$ -equivariant and  $U$  is also invariant by  $R_{\frac{1}{m}}$ . Hence, the following holds:

$$f^* \circ R_{\frac{1}{m}}^* = R_{\frac{1}{m}}^* \circ f^*.$$

Since  $R_{\frac{1}{m}}^*$  is a periodic homeomorphism of  $\mathbf{T}^1$  with rotation number  $1/m$ , then  $R_{\frac{1}{m}}^*$  is conjugated to the linear rotation  $R_{\frac{1}{m}}$  and  $f^*$  is said to be  $\mathbf{Z}_m$ -equivariant in the space of prime ends.

**Theorem 3 (Theorem 4.2 in [1]).** *Given an irrational number  $\tau \notin \mathbf{Q}$ , there exists a  $\mathbf{Z}_m$ -equivariant and admissible homeomorphism in  $\mathbf{R}^2$  with rotation number  $\bar{\tau} \in$*

**T** and such that induces a Denjoy map in the circle of prime ends which is also  $\mathbf{Z}_m$ -equivariant.

Hence, [1] shows that for  $\mathbf{Z}_m$ -equivariant homeomorphisms one cannot guarantee that the rotation number is rational and proves the existence of  $\mathbf{Z}_m$ -equivariant homeomorphisms with some complicated and interesting dynamical features.

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